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10

A remark on characters of unitary representations of semi-simple Lie groups

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Suppose we have a real semi-simple Lie group G and an irreducible unitary representation $(\rho, \mathfrak{H})$ of G. It is then well known that the 'trace' of the representation can be defined in a natural way to be a distribution (and hence a hyperfunction) on G, and is called the character of the representation. We denote the character by χ.

The purpose of this note is to show that, if K is a maximal compact subgroup of G, the restriction $\chi|K$ of χ does make sense to be a hyperfunction (in fact a distribution) on K in a very natural way, and that the fact is an easy corollary of a general result in hyperfunction theory.

For the reader's convenience we first quote some of general results in the theory of hyperfunctions ([1],[2],[3] and [4]).

Let M be a real analytic manifold, $\mathcal{A}_M$ be a sheaf of germs of real analytic functions. Then we can define the sheaf of germs of hyperfunctions over M, denoted by $\mathcal{B}_M$. $\mathcal{B}_M$ satisfies following properties;

i) $\mathcal{B}_M$ is a left $\mathcal{A}_M$-Module. Moreover, if we denote by $\mathcal{D}_M$ the shaf of rings of germs of linear differential operators of finite order with real analytic coefficients, then $\mathcal{B}_M$ is a left $\mathcal{D}_M$-module.

ii) There is a canonical $\mathcal{D}_M$-linear injection $\alpha: \mathcal{A}_M \to \mathcal{B}_M$

iii) $\mathcal{B}_M$ is a flabby sheaf.

iv) $\mathcal{B}_M$ contains a sheaf of germs of distributions in the sense of L. Schwartz.

Let T^*M be a cotangent bundle of M, $S^*M=(T^*M-M)/\mathbb{R}^+$ be the sphere bundle corresponding to T^*M, called cotangential sphere bundle of M, where $\mathbb{R}^+$ is a multiplicative group of positive real numbers. Fixing a local coordinate of M, the point of T^*M is represented by (x,η), where x is a coordinate of M, and η is a cotangent vector. In this notation, we denote by $(x, \sqrt{-1}\eta\infty)$ the corresponding point in S^*M. We denote by $\pi: S^*M \to M$ the natural projection. We can construct the sheaf $\mathcal{C}_M$ of S^*M. $\mathcal{C}_M$ is a sheaf describing singularities of hyperfunctions. $\mathcal{C}_M$ has following properties;

v) $\mathcal{C}_M$ is a left $\pi^{-1}\mathcal{A}_M$-Module. Moreover, $\mathcal{C}_M$ is a left $\pi^{-1}\mathcal{D}_M$-Module.

vi) There is a canonical $\mathcal{D}_M$-linear homomorphism β: $\mathcal{B}_M \to \pi_* \mathcal{C}_M$, such that

$$0 \to \mathcal{A}_M \xrightarrow{\alpha} \mathcal{B}_M \xrightarrow{\beta} \pi_* \mathcal{C}_M \to 0$$

is exact.

vii) $\mathcal{C}_M$ is a flabby sheaf.

Let $u(x)$ be a hyperfunction on M. We denote by $\text{sing.supp.}_M(u)$ the smallest closed subset of M such that u is real analytic in its complementary set. The support of $\beta(u) \in \Gamma(S^*M; \mathcal{C}_M)$ is denoted by $S\text{-}S(u)$, which is a closed subset of S^*M. By vi), we have $\pi(S\text{-}S(u)) = \text{sing. supp}(u)$.

We use following two theorems. Let N be a real analytic submanifold of M. T_N^*M is a conormal bundle of N, that is, the kernel of $N \underset{M}{\times} T^*M \to T^*N$. We put $S_N^*M = (T_N^*M - N)/\mathbb{R}^+$. S_N^*M is a closed subset of S^*M.

<u>Theorem A</u> Let $u(x)$ be a hyperfunction on M satisfying $S - S(u) \cap S_N^*M = \phi$. Then we can canonically define the restriction $u|_N$ of u to N. $u|_N$ is a hyperfunction on N.

<u>Theorem B</u> Let $P(x, D_x)$ be a linear differential operator with real analytic coefficients of order m. Let $\sigma(P)$ be a principal symbol of P, which is a function on T^*M, homogeneous of degree m. We put

$$F = \{(x, \sqrt{-1}\eta\infty) \in S^*M;\ \sigma(P)(x,\eta) = 0\}$$

This is a closed set of S^*M. Then, for every hyperfunction $u(x)$ on M, we have $S - S(u) \subset F \cup S - S(Pu)$. Especially, if Pu is a real analytic function, then we have $S\text{-}S(u) \subset F$.

Now, we apply the preceding theory to the theory of unitary representation.

Let G be a real semi-simple Lie group, $\mathfrak{g}$ be a Lie algebra of G. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, K be a compact subgroup of G corresponding to the compact Lie subalgebra $\mathfrak{k}$. It is well known that the Killing form is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. Let C be the Casimir operator of G, which is a bi-invariant linear differential operator on G of order 2. Let $(U, \mathfrak{H})$ be an irreducible unitary representation of G. χ be its character. By the result of Harish-Chandra, χ is a distribution (and hence a hyperfunction) on G, satisfying;

a) $\chi(gxg^{-1}) = \chi(x)$ for any $(g,x) \in G \times G$

b) $C\chi = a\chi$ for some $a \in \mathbb{C}$

Harish-Chandra has also proved that unitary representation is uniquely determined by its character.

Our purpose is to show the following theorem;

Theorem Let $\chi(x)$ be a hyperfunction on G satisfying; (b) $C\chi(x) = a\chi(x)$ for some $a \in \mathbb{C}$. Then we can canonically define the restriction $\chi|K$ of χ to the maximal compact subgroup K of G, which is a hyperfunction on K.

Proof

Let e be the neutral element of $\mathfrak{g}$. Then $T^*G_e = \mathfrak{g}^*$, $T^*_K G_e = \mathfrak{p}^* \subset \mathfrak{g}^*$. Let $\sigma(C)$ be a principal symbol of C, which is bi-invariant. At e, $\sigma(C)$ can be considered as a quadratic form on $\mathfrak{g}^*$, which coincides with the dual form of the Killing form. Therefore $\sigma(C)$ is positive definite on $\mathfrak{p}^*$. It follows that $\sigma(C)$ never vanishes on $T^*_K G_e$. Since $\sigma(C)$ is bi-invariant, $\sigma(C)$ vanishes nowhere on $T^*_K G$. By the theorem B, $S - S(\chi) \cap S^*_K G = \phi$. Using the theorem A, $\chi|K$ can be defined. q. e. d.

Remark.

Recently, L. Hörmander has proved an analogue of the theorem A and the theorem B in the category of distribution ([5]). Therefore, in Theorem, if χ is a distribution satisfying (b), $\chi|K$ is also a distribution on K.

Reference

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